BOSONIC FORMULA FOR LEVEL-RESTRICTED PATHS

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Abstract. We prove a bosonic formula for the generating function of level-restricted paths for the infinite families of affine Kac-Moody algebras. In affine type A this yields an expression for the level-restricted generalized Kostka polynomials.

1. Introduction

Let $\mathfrak{g}$ be an affine Kac-Moody algebra, V a $U_q(\mathfrak{g})^+$-submodule of a finite direct sum V' of irreducible integrable highest weight $U_q(\mathfrak{g})$-modules, and Π the limit of the Demazure operator for an element w of the Weyl group as $\ell(w) \to \infty$. The main theorem of this paper gives sufficient conditions on V so that the formula

$$\Pi \ \mathrm{ch}(V) = \mathrm{ch}(V') \tag{1.1}$$

holds, where $\mathrm{ch}(V)$ is the character of V. When V is the one-dimensional $U_q(\mathfrak{g})^+$-module generated by the dominant integral weight Λ then (1.1) is the Weyl-Kac character formula. The above result is well-known when V is a union of Demazure modules for any Kac-Moody algebra $\mathfrak{g}$.

Let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$. Consider the $\mathfrak{g}'$-module V given by a tensor product of finite-dimensional $U_q(\mathfrak{g}')$-modules that admit a crystal of level at most ℓ, with the one-dimensional subspace generated by a highest weight vector of an irreducible integrable highest weight $U_q(\mathfrak{g}')$-module of level ℓ. Such modules V can be given the structure of a $U_q(\mathfrak{g})^+$-module and as such, satisfy the above conditions. Then a special case of (1.1) is a bosonic formula for the q-enumeration of level-restricted inhomogeneous paths by the energy function. In type $A_{n-1}^{(1)}$ this formula was conjectured in [3], stated there as a q-analogue of the Goodman-Wenzl straightening algorithm for outer tensor products of irreducible modules over the type A Hecke algebra at a root of unity [4]. In the isotypic component of the vacuum, the bosonic formula coincides with half of the bose-fermi conjecture in [20, (9.2)].

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2. Notation

Most of the following notation is taken from ref. [7]. Let X be a Dynkin diagram of affine type with vertices indexed by the set $I = \{0, 1, 2, \ldots, n\}$ as in [7], Cartan matrix $A = (a_{ij})_{i,j\in I}$, $\mathfrak{g} = \mathfrak{g}(A)$ the affine Kac-Moody algebra, and $\mathfrak{h}$ the Cartan subalgebra. Let $\{\alpha_i^\vee : i \in I\} \subset \mathfrak{h}$ and $\{\alpha_j : j \in I\} \subset \mathfrak{h}^*$ be the simple coroots and roots, which are linearly independent subsets that satisfy $\langle \alpha_i^\vee, \alpha_j\rangle = a_{ij}$ for $i, j \in I$ where $\langle\cdot,\cdot\rangle : \mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{C}$ is the natural pairing. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the root lattice. Let the null root $\delta = \sum_{i\in I} a_i\alpha_i$ be the unique element of the positive cone $\bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i$ in Q, that generates the one-dimensional lattice $\{\beta \in Q | \langle \alpha_i^\vee, \beta\rangle = 0 \text{ for all } i \in I\ \}$. Let $K = \sum_{i\in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$ be the canonical central element, where the integers $a_i^\vee$ are the analogues of the integers a_i for the dual algebra $\mathfrak{g}^\vee$ defined by the transpose ${}^tA$ of the Cartan matrix A. Let $d \in \mathfrak{h}$ (the degree derivation) be defined by the conditions $\langle d, \alpha_i\rangle = \delta_{i0}$ where δ_{ij} is the Kronecker delta; d is well-defined up to a summand proportional to K. Then $\{\alpha_0^\vee, \ldots, \alpha_n^\vee, d\}$ is a basis of $\mathfrak{h}$. Let $\{\Lambda_0, \ldots, \Lambda_n, \delta\}$ be the dual basis of $\mathfrak{h}^*$; the elements $\{\Lambda_0, \ldots, \Lambda_n\}$ are called the fundamental weights. The weight lattice is defined by $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \bigoplus \mathbb{Z}a_0^{-1}\delta$; in the usual definition the scalar a_0^{-1} is absent. The weight lattice contains the root lattice since $\alpha_j = \sum_{i\in I} a_{ij}\Lambda_i$ for $j \in I$. Define $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \Lambda_i \bigoplus \mathbb{Z}a_0^{-1}\delta$. Say that a weight $\Lambda \in P^+$ has level ℓ if $\ell = \langle K, \Lambda\rangle$.

Consider the standard symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. Since $\{\alpha_0, \ldots, \alpha_n, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$, this form is uniquely defined by $(\alpha_i|\alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ for $i, j \in I$, $(\alpha_i|\Lambda_0) = \delta_{i0}a_0^{-1}$ for $i \in I$ and $(\Lambda_0|\Lambda_0) = 0$. This form induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by $a_i^\vee \nu(\alpha_i^\vee) = a_i\alpha_i$ for $i \in I$ and $\nu(d) = a_0\Lambda_0$. Also $\nu(K) = \delta$.

The Weyl group W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections r_i $(i \in I)$ defined by

$$r_i(\beta) = \beta - \langle \alpha_i^\vee, \beta\rangle\alpha_i.$$

The form $(\cdot|\cdot)$ is W-invariant. Suppose $\alpha \in Q$ is a real root, that is, the α-weight space of $\mathfrak{g}$ is nonzero and there is a simple root α_i and a Weyl group element $w \in W$ such that $\alpha = w(\alpha_i)$. Define $\alpha^\vee \in \mathfrak{h}$ by $w(\alpha_i^\vee)$. This is independent of the expression $\alpha = w(\alpha_i)$. Define $r_\alpha \in W$ by

$$r_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta\rangle\alpha \qquad \text{for } \beta \in \mathfrak{h}^*.$$

Let $\mathfrak{g}'$ be the derived algebra of $\mathfrak{g}$, obtained by "omitting" the degree derivation d. Its weight lattice is $P_{cl} \cong P/\mathbb{Z}a_0^{-1}\delta$. Denote the canonical projection $P \to P_{cl}$ by cl. Write $\alpha_i^{cl} = \mathrm{cl}(\alpha_i)$ and $\Lambda_i^{cl} = \mathrm{cl}(\Lambda_i)$ for $i \in I$. The elements $\{\alpha_i^{cl} \mid i \in I\}$ are linearly dependent. Write $\mathrm{af} : P_{cl} \to P$ for the section of cl given by $\mathrm{af}(\Lambda_i^{cl}) = \Lambda_i$ for all $i \in I$. Write $P_{cl}^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \Lambda_i^{cl}$. Define the level of $\mu \in P_{cl}^+$ to be $\langle K, \mathrm{af}(\mu)\rangle$.

Consider the Dynkin diagram $\overline{X}$ obtained by removing the vertex 0 from the diagram X, with corresponding Cartan matrix $\overline{A}$ indexed by the set $J = I - \{0\}$, and let $\overline{\mathfrak{g}} = \mathfrak{g}(\overline{A})$ be the simple Lie algebra. One has the inclusions $\overline{\mathfrak{g}} \subset \mathfrak{g}' \subset \mathfrak{g}$. Let $\{\overline{\alpha}_i : i \in J\}$ be the simple roots, $\{\overline{\Lambda}_i : i \in J\}$ the fundamental weights, and $\overline{Q} = \bigoplus_{i\in J} \mathbb{Z}\overline{\alpha}_i$ the root lattice for $\overline{\mathfrak{g}}$. The weight lattice of $\overline{\mathfrak{g}}$ is $\overline{P} = \bigoplus_{i\in J} \mathbb{Z}\overline{\Lambda}_i$ and $\overline{P} \cong P_{cl}/\mathbb{Z}\Lambda_0$. The image of $\Lambda \in P$ into $\overline{P}$ is denoted by $\overline{\Lambda}$. We shall use the section of the natural projection $P_{cl} \to \overline{P}$ given by the map $\overline{P} \to P_{cl}$ that sends $\overline{\Lambda}_i \mapsto \Lambda_i^{cl} - \Lambda_0^{cl}$ for $i \in J$. By abuse of notation, for $\Lambda \in P$, $\overline{\Lambda}$ shall also denote the image of the element $\overline{\Lambda}$ under the lifting map $\overline{P} \to P$ specified above.

Let $\overline{P}^+ = \bigoplus_{i \in J} \mathbb{Z}_{\geq 0} \overline{\Lambda}_i$. For $\lambda \in \overline{P}^+$, denote by $V(\lambda)$ the irreducible integrable highest weight $U_q(\overline{\mathfrak{g}})$-module of highest weight λ.

Let $\theta = \delta - a_0\alpha_0 = \sum_{i \in J} a_i\alpha_i \in \overline{Q}$. One has the formulas $(\theta|\theta) = 2a_0$, $\theta = a_0\nu(\theta^\vee)$, and $\alpha_0^\vee = K - a_0\theta^\vee$. Observe that

$$\mathrm{cl}(\alpha_0) = -a_0^{-1} \sum_{i \in J} a_i \alpha_i^{cl} = -\mathrm{cl}(\nu(\theta^\vee)).$$

For $\Lambda \in P^+$ let $\mathbb{V}(\Lambda)$ be the irreducible integral highest weight module of highest weight Λ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$, $\mathbb{B}(\Lambda)$ the crystal base of $\mathbb{V}(\Lambda)$, and $u_\Lambda \in \mathbb{B}(\Lambda)$ the highest weight vector.

By restriction from $U_q(\mathfrak{g})$ to $U_q(\mathfrak{g}')$, the module $\mathbb{V}(\Lambda)$ is an irreducible integral highest weight module for $U_q(\mathfrak{g}')$ of highest weight $\mathrm{cl}(\Lambda)$, with crystal $\mathbb{B}(\Lambda)$ that is P_{cl}-weighted by composing the weight function $\mathbb{B}(\Lambda) \to P$ with the projection cl. Conversely, any integrable irreducible highest weight $U_q(\mathfrak{g}')$-module can be obtained this way.

3. Short review of affine crystal theory

3.1. **Crystals.** A P-weighted I-crystal B is a colored graph with vertices indexed by $b \in B$, directed edges colored by $i \in I$, and a weight function $\mathrm{wt} : B \to P$, satisfying the axioms below. First some notation is required. Denote an edge from b to b' colored i, by $b' = f_i(b)$ or equivalently $b = e_i(b')$. Write $\phi_i(b)$ (resp. $\epsilon_i(b)$) for the maximum index m such that $f_i^m(b)$ (resp. $e_i^m(b)$) is defined.

1. If $b' = f_i(b)$ then $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.
2. $\phi_i(b) - \epsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$.

An element $u \in B$ is a highest weight vector if $e_i(u)$ is undefined for all $i \in I$. The i-string of $b \in B$ consists of all elements $e_i^m(b)$ $(0 \leq m \leq \epsilon_i(b))$ and $f_i^m(b)$ $(0 \leq m \leq \phi_i(b))$. The nondominant part of the i-string is comprised of all elements which admit e_i.

We also define the crystal reflection operator $s_i : B \to B$ by

$$s_i(b) = \begin{cases} f_i^{\phi_i(b)-\epsilon_i(b)}(b) & \text{if } \phi_i(b) > \epsilon_i(b) \\ b & \text{if } \phi_i(b) = \epsilon_i(b) \\ e_i^{\epsilon_i(b)-\phi_i(b)}(b) & \text{if } \phi_i(b) < \epsilon_i(b). \end{cases}$$

It is obvious that s_i is an involution. Observe that

$$\mathrm{wt}(s_i(b)) = r_i \mathrm{wt}(b) = \mathrm{wt}(b) - \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \alpha_i. \tag{3.1}$$

Define the notation $\phi(b) = \sum_{i \in I} \phi_i(b)\Lambda_i$ and $\epsilon(b) = \sum_{i \in I} \epsilon_i(b)\Lambda_i$.

If a $U_q(\mathfrak{g})$-module (resp. $U_q(\mathfrak{g}')$-module, resp. $U_q(\overline{\mathfrak{g}})$-module) has a crystal base then the latter is naturally a P-weighted (resp. P_{cl}-weighted, resp. $\overline{P}$-weighted) I-crystal (resp. I-crystal, resp. J-crystal).

3.2. **Tensor products.** Given crystals B_1 and B_2, contrary to the literature (but consistent with the Robinson-Schensted-Knuth correspondence in type A), define the following crystal structure on the tensor product $B_2 \otimes B_1$. The elements are denoted $b_2 \otimes b_1$ for $b_i \in B_i$ $(i \in \{1,2\})$ and one defines

$$\begin{aligned} \phi_i(b_2 \otimes b_1) &= \phi_i(b_2) + \max(0, \phi_i(b_1) - \epsilon_i(b_2)) \\ \epsilon_i(b_2 \otimes b_1) &= \epsilon_i(b_1) + \max(0, -\phi_i(b_1) + \epsilon_i(b_2)). \end{aligned}$$

When $\phi_i(b_2 \otimes b_1) > 0$ (resp. $\epsilon_i(b_2 \otimes b_1) > 0$) one defines

$$f_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes f_i(b_1) & \text{if } \phi_i(b_1) > \epsilon_i(b_2) \\ f_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) \leq \epsilon_i(b_2) \end{cases}$$

and respectively

$$e_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes e_i(b_1) & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2) \\ e_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) < \epsilon_i(b_2). \end{cases}$$

An element of a tensor product of crystals is called a path.

3.3. **Energy function.** The definitions here follow [16]. Suppose that B_1 and B_2 are crystals of finite-dimensional $U_q(\mathfrak{g}')$-modules such that $B_2 \otimes B_1$ is connected. Then there is an isomorphism of P_{cl}-weighted I-crystals $B_2 \otimes B_1 \cong B_1 \otimes B_2$. This is called the local isomorphism. Let the image of $b_2 \otimes b_1 \in B_2 \otimes B_1$ under this isomorphism be denoted $b_1' \otimes b_2'$. Then there is a unique (up to a global additive constant) map $H : B_2 \otimes B_1 \to \mathbb{Z}$ such that

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = e_0(b_2) \otimes b_1 \\ & \text{and } e_0(b_1' \otimes b_2') = e_0(b_1') \otimes b_2', \\ 1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = b_2 \otimes e_0(b_1) \\ & \text{and } e_0(b_1' \otimes b_2') = b_1' \otimes e_0(b_2'), \\ 0 & \text{otherwise.} \end{cases}$$

This map is called the local energy function.

Consider $B = B_L \otimes \cdots \otimes B_1$ with B_j the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module for $1 \leq j \leq L$. Assume that for all $1 \leq i < j \leq L$, $B_j \otimes B_i$ is a connected P_{cl}-weighted I-crystal. Given $b = b_L \otimes \cdots \otimes b_1 \in B$, denote by $b_j^{(i+1)}$ the $(i+1)$-th tensor factor in the image of b under the composition of local isomorphisms that switch B_j with B_k as k goes from $j-1$ down to $i+1$. Then define the energy function

$$E_B(b) = \sum_{1 \leq i < j \leq L} H_{j,i}(b_j^{(i+1)} \otimes b_i) \tag{3.2}$$

where $H_{j,i} : B_j \otimes B_i \to \mathbb{Z}$ is the local energy function. It satisfies the following property.

Lemma 1. [5, Prop. 1.1] *Suppose $i \in I$, $b \in B$ and $e_i(b)$ is defined. If $i \neq 0$ then $E_B(e_i(b)) = E_B(b)$. If $i = 0$ and b has the property that for any of its images $b' = b_L' \otimes \cdots \otimes b_1'$ under a composition of local isomorphisms, $e_0(b') = b_L' \otimes \cdots \otimes e_0(b_k') \otimes \cdots \otimes b_1'$ with $k \neq 1$, then $E_B(e_0(b)) = E_B(b) - 1$.*

3.4. **Classically restricted paths.** Say that $b \in B := B_L \otimes \cdots \otimes B_1$ is classically restricted if b is a $\overline{\mathfrak{g}}$-highest weight vector, that is, $e_i(b)$ is undefined for all $i \in J$. For $\lambda \in \overline{P}^+$ denote by $\mathcal{P}(B, \lambda)$ the set of classically restricted $b \in B$ of weight λ. Define the polynomial

$$K(B, \lambda)(q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{E_B(b)} \tag{3.3}$$

where E_B is the energy function on B. For $\mathfrak{g}$ of type $A_{n-1}^{(1)}$ $K(B, \lambda)(q)$ is the generalized Kostka polynomial [18, 19, 20].

3.5. **Almost perfect crystals.** Let B be the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module. Say that B is almost perfect of level ℓ [17] if it satisfies the following weakening of the definition of a perfect crystal [9, Def. 4.6.1]:

1. $B \otimes B$ is connected.
2. There is a $\Lambda' \in P_{cl}$ such that there is a unique $b' \in B$ such that $\mathrm{wt}(b') = \Lambda'$ and for every $b \in B$, $\mathrm{wt}(b) \in \Lambda' - \bigoplus_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i$.
3. For every $b \in B$, $\langle K, \epsilon(b)\rangle \geq \ell$.
4. For every $\Lambda \in P_{cl}^+$ of level ℓ, there is a $b, b' \in B$ such that $\epsilon(b) = \phi(b') = \Lambda$.

B is said to be perfect if the elements b and b' in item 4 are unique.

3.6. **Level restricted paths.** From now on, fix a positive integer ℓ (the level).

For $1 \leq j \leq L$ let B_j be the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module, that is almost perfect of level at most ℓ.

Let $B = B_L \otimes \cdots \otimes B_1$, $\Lambda, \Lambda' \in P_{cl}^+$ weights of level ℓ, and $\mathcal{P}(B, \Lambda, \Lambda')$ the set of paths $b = b_L \otimes \cdots \otimes b_1 \in B$ such that $b \otimes u_\Lambda \in B \otimes \mathbb{B}(\Lambda)$ is a highest weight vector of weight Λ'.

In the special case that $\Lambda = \ell\Lambda_0$, the elements of $\mathcal{P}(B, \Lambda, \Lambda')$ are called the level-ℓ restricted paths of weight Λ'.

Theorem 2. [9] [13, Appendix A]. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra in one of the infinite families. Let B be the tensor product of crystals of finite-dimensional $U_q(\mathfrak{g}')$-modules, that are almost perfect of level at most ℓ, and $\Lambda \in P_{cl}^+$ a weight of level ℓ. Then there is an isomorphism of P_{cl}-weighted I-crystals*

$$B \otimes \mathbb{B}(\Lambda) \cong \bigoplus_{\Lambda' \in P_{cl}^+} \bigoplus_{b \in \mathcal{P}(B,\Lambda,\Lambda')} \mathbb{B}(\Lambda') \tag{3.4}$$

where Λ' is of level ℓ.

This isomorphism of P_{cl}-weighted crystals can be lifted to one of P-weighted crystals by specifying an integer multiple of $a_0^{-1}\delta$ for each highest weight vector in $B \otimes \mathbb{B}(\Lambda)$. However for our purposes this should be done in a way that extends the definition of the energy function for B. To this end, choose a perfect crystal B_0 of level ℓ, and assume that for all $0 \leq i < j \leq L$, $B_j \otimes B_i$ is connected. Let $b_0 \in B_0$ be the unique element such that $\phi(b_0) = \Lambda$. Define the energy function $E : B \to \mathbb{Z}$ by $E(b) = E_{B,B_0}(b \otimes b_0)$ where $E_{B,B_0} : B \otimes B_0 \to \mathbb{Z}$ is the energy function defined in (3.2). For $b \in \mathcal{P}(B, \Lambda, \Lambda')$, define an affine weight function $\mathrm{wt}(b \otimes u_\Lambda) = \mathrm{af}(\Lambda') - E(b)a_0^{-1}\delta$. This defines the P-weight of every highest weight vector in $B \otimes \mathbb{B}(\Lambda)$ and hence a P-weight function for all of $B \otimes \mathbb{B}(\Lambda)$.

Then one has the following P-weighted analogue of (3.4):

$$B \otimes \mathbb{B}(\mathrm{af}(\Lambda)) \cong \bigoplus_{\Lambda' \in P_{cl}^+} \bigoplus_{b \in \mathcal{P}(B,\Lambda,\Lambda')} \mathbb{B}(\mathrm{wt}(b \otimes u_\Lambda)) \tag{3.5}$$

where Λ' is of level ℓ. This decomposition can be described by the polynomial

$$K(B, \Lambda, \Lambda', B_0)(q) = \sum_{b \in \mathcal{P}(B,\Lambda,\Lambda')} q^{E(b)}. \tag{3.6}$$

Our goal is to prove a formula for the polynomial $K(B, \Lambda, \Lambda', B_0)(q)$.

4. General bosonic formula

Let J be the antisymmetrizer

$$J = \sum_{w \in W} (-1)^w w.$$

Write

$$R = \prod_{\alpha \in \Delta_+} (1 - \exp(-\alpha))^{\mathrm{mult}(\alpha)}$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\mathrm{mult}(\alpha)$ is the dimension of the α-weight space in $\mathfrak{g}$.

Let $\rho \in P^+$ be the unique weight defined by $\langle \alpha_i^\vee, \rho \rangle = 1$ for all $i \in I$ and $\langle d, \rho \rangle = 0$. It satisfies $\langle \theta^\vee, \rho \rangle = a_0^{-1} \langle K - \alpha_0^\vee, \rho \rangle = a_0^{-1}(h^\vee - 1)$ where $h^\vee = \sum_{i \in I} a_i^\vee$ is the dual Coxeter number. Define the operator

$$\Pi(p) = R^{-1} e^{-\rho} J(e^\rho p).$$

where R^{-1} makes sense by expanding the reciprocals of the factors of R in geometric series. The computation is defined in a suitable completion of $\mathbb{Z}[P]$. One has $\Pi(e^\Lambda) = \mathrm{ch}\, \mathbb{V}(\Lambda)$ for all $\Lambda \in P^+$, which is the Weyl-Kac character formula [7, Theorem 10.4].

Theorem 3. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra in one of the infinite families, B' the crystal of a finite direct sum of irreducible integrable highest weight $U_q(\mathfrak{g})$-modules and $B \subset B'$ a subset such that:*

1. *B is closed under e_i for all $i \in I$.*
2. *B' is generated by B.*
3. *For all $b \in B$ and $i \in I$, if $\epsilon_i(b) > 0$ then the i-string of b in B' is contained in B.*

Then

$$\Pi\, \mathrm{ch}(B) = \mathrm{ch}(B'). \tag{4.1}$$

Proof. Without loss of generality it may be assumed that $B' = \mathbb{B}(\Lambda)$ for some $\Lambda \in P^+$. Multiplying both sides of (4.1) by $R\, e^\rho$, one obtains

$$\sum_{(w,b) \in W \times B} (-1)^w w(e^{\mathrm{wt}(b) + \rho}) = \sum_{w \in W} (-1)^w w(e^{\Lambda + \rho}).$$

Observe that both sides are W-alternating. The W-alternants have a basis given by $J(\Lambda + \rho)$ where $\Lambda \in P^+$. Taking the coefficient of $e^{\Lambda+\rho}$ on both sides,

$$\sum_{(w,b) \in \mathcal{S}} (-1)^w = 1 \tag{4.2}$$

where $\mathcal{S}$ is the set of pairs $(w, b) \in W \times B$ such that

$$\mathrm{wt}(b) = w^{-1}(\Lambda + \rho) - \rho. \tag{4.3}$$

Observe that if $(w, b) \in \mathcal{S}$ is such that b is a highest weight vector, then $w = 1$ and $b = u_\Lambda$, for both of the regular dominant weights $\mathrm{wt}(b) + \rho$ and $\Lambda + \rho$ are in the same W-orbit and hence must be equal. Conditions 1 and 2 ensure that $u_\Lambda \in B$. Let $\mathcal{S}' = \mathcal{S} - \{(1, u_\Lambda)\}$. It is enough to show that there is an involution $\Phi : \mathcal{S}' \to \mathcal{S}'$ with no fixed points, such that if $\Phi(w, b) = (w', b')$ then w and w' have opposite sign. In this case Φ is said to be sign-reversing. Let $\mathcal{S}_i$ be the set of pairs $(w, b) \in \mathcal{S}'$

such that $\epsilon_i(b) > 0$. Define the map $\Phi_i : \mathcal{S}_i \to \mathcal{S}_i$ by $\Phi_i(w,b) = (wr_i, s_ie_i(b))$. Note that $s_ie_i(b) \in B$ by condition 3. The condition (4.3) for $\Phi_i(w,b)$ is

$$\begin{aligned}(wr_i)^{-1}(\Lambda+\rho)-\rho &= r_iw^{-1}(\Lambda+\rho) - r_i\rho + r_i\rho - \rho \\ &= r_i(w^{-1}(\Lambda+\rho)-\rho) - \langle \alpha_i^\vee, \rho\rangle \alpha_i \\ &= r_i(\mathrm{wt}(b)) - \alpha_i = \mathrm{wt}(f_is_i(b)) = \mathrm{wt}(s_ie_i(b)).\end{aligned}$$

Since $s_ie_i(b) = f_is_i(b)$, $\epsilon_i(s_ie_i(b)) > 0$, so that $(wr_i, s_ie_i(b)) \in \mathcal{S}_i$. This shows that Φ_i is well-defined. It follows directly from the definitions that Φ_i is a sign-reversing involution.

Since $\mathcal{S}' = \bigcup_{i\in I} \mathcal{S}_i$ it suffices to define a global involutive choice of the canceling root direction for each pair $(w,b) \in \mathcal{S}'$, that is, a function $v : \mathcal{S}' \to I$ such that if $v(w,b) = i$ then

(V1) $(w,b) \in \mathcal{S}_i$.
(V2) $v(wr_i, s_ie_i(b)) = i$.

Let $\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_\ell}$ be an expression of Λ as a sum of fundamental weights. By [6, Lemma 8.3.1], $\mathbb{B}(\Lambda)$ is isomorphic to the full subcrystal of $\mathbb{B}(\Lambda_{i_\ell}) \otimes \cdots \otimes \mathbb{B}(\Lambda_{i_1})$ generated by $u_{\Lambda_{i_\ell}} \otimes \cdots \otimes u_{\Lambda_{i_1}}$.

Given $(w,b) \in \mathcal{S}'$, let $b_\ell \otimes \cdots \otimes b_1$ be the image of b in the above tensor product of crystals of modules of fundamental highest weight. Let r be minimal such that $b_r \otimes b_{r-1} \otimes \cdots \otimes b_1$ is not a highest weight vector. Then $b_{r-1} \otimes \cdots \otimes b_1$ is a highest weight vector, say of weight Λ'.

Let $\mathcal{B}$ be a perfect crystal of the same level as Λ_{i_r}. Given any $L > 0$, the theory of perfect crystals [9, Section 4.5] gives an isomorphism of P-weighted crystals

$$\mathbb{B}(\Lambda_{i_r}) \cong \mathcal{B}^{\otimes L} \otimes \mathbb{B}(\Lambda_j)$$

where j is determined by i_r and L and $\mathcal{B}^{\otimes L}$ is P-weighted using the energy function.

Let $b_r \in \mathbb{B}(\Lambda_{i_r})$ have image $p_{-1} \otimes \cdots \otimes p_{-L} \otimes u'$ where $u' \in \mathbb{B}(\Lambda_j)$. Assume that L is large enough so that $u' = u_{\Lambda_j}$. If one takes the image of b_r in such a tensor product for $L' > L$ the tensor factors p_{-1} through p_{-L} do not change.

Let k be minimal such that $p_k \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is not a highest weight vector. Observe that k is independent of L as long as L is big enough. Then $p_{k-1} \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is a highest weight vector, say of weight Λ''.

So $p_k \in \mathcal{B}$ is such that $\epsilon_i(p_k) > \langle \alpha_i^\vee, \Lambda''\rangle$ for some $i \in I$; let I' be the set of such $i \in I$.

Fix an $i \in I'$. Consider the same constructions for $b' = s_ie_i(b)$. Let $b'_\ell \otimes \cdots \otimes b'_1$ be the image of b' in the above tensor product of irreducible crystals of fundamental highest weights. Then $b'_{r-1} \otimes \cdots \otimes b'_1 = b_{r-1} \otimes \cdots \otimes b_1$ and $b'_r \otimes \cdots \otimes b'_1$ is not a highest weight vector; in particular it admits e_i. Take L large enough so that the image of b'_r in $\mathcal{B}^{\otimes L} \otimes \mathbb{B}(\Lambda_j)$ also has the form $p'_{-1} \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j}$. Then $p_{k-1} \otimes \cdots \otimes p_{-L} = p'_{k-1} \otimes \cdots \otimes p'_{-L}$ and $p'_k \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ admits e_i.

The level of the fundamental weight Λ_i is $a_i^\vee$. For the affine algebras $A_n^{(1)}$ and $C_n^{(1)}$, $a_i^\vee = 1$ for all $i \in I$. For all others $1 \le a_i^\vee \le 2$. The theorem now follows from Lemma 4 below, applied with the dominant integral weight Λ''. □

Lemma 4. *For the affine Kac-Moody algebra $\mathfrak{g}$ in one of the infinite families, there exist perfect crystals B of level one and two (the latter case only for $\mathfrak{g} \neq A_n^{(1)}, C_n^{(1)}$) having the following property. Let Λ be a dominant integral weight of positive level,*

S the set of elements $b_1 \in B$ such that $b_1 \otimes u_\Lambda$ is not a highest weight vector. Then there is a map $v : S \to I$ such that if $v(b_1) = i$ then

1. *$\epsilon_i(b_1 \otimes u_\Lambda) > 0$.*
2. *For any $b_L, \dots, b_2 \in B$, writing $b'_L \otimes \cdots \otimes b'_2 \otimes b'_1 \otimes u_\Lambda = s_i e_i(b_L \otimes \cdots \otimes b_2 \otimes b_1 \otimes u_\Lambda)$, one has $b'_1 \in S$ and $v(b'_1) = i$.*

Proof. For the involutive property 2, it is sufficient that v is constant on the nondominant part of every string. Hence one only needs to consider

(4.4) $\quad b_1$ that are on the nondominant part of at least two strings of length ≥ 2.

Perfect crystals of level one for $A_n^{(1)}$ ($n \geq 1$), $B_n^{(1)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$) and $D_{n+1}^{(2)}$ ($n \geq 2$) are listed in Table 1 (see [9, Section 6]). Note that there are no elements satisfying (4.4). This guarantees the existence of the map v with the desired properties.

The crystal $B(2\Lambda_1) \oplus B(0)$ is a level one perfect crystal for $C_n^{(1)}$ ($n \geq 2$) [8]. The crystal graph corresponding to the integrable highest weight module $V(\Lambda_1)$ of $U_q(C_n)$ is given by [14, (4.2.4)]

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}.$$

The crystal $B(2\Lambda_1)$ is the connected component of $B(\Lambda_1) \otimes B(\Lambda_1)$ containing $u_{\Lambda_1} \otimes u_{\Lambda_1}$ (see [14, Section 4.4]) which fixes the action of e_i and f_i for $1 \leq i \leq n$. The edges in $B(2\Lambda_1) \oplus B(0)$ corresponding to f_0 are given by [8]

$$\begin{array}{lll}
\boxed{i}\boxed{\bar{1}} \xrightarrow{0} \boxed{1}\boxed{i} & & \text{for } i \neq 1, \bar{1} \\
\boxed{\bar{1}}\boxed{\bar{1}} \xrightarrow{0} \emptyset & & \\
\emptyset \xrightarrow{0} \boxed{1}\boxed{1}. & &
\end{array}$$

There are the following strings of length greater than one

$$\begin{array}{llllll}
 & \boxed{k}\boxed{k} & \xrightarrow{k} \boxed{k}\boxed{k+1} & \xrightarrow{k} \boxed{k+1}\boxed{k+1} & & \text{for } 1 \leq k < n \\
(4.5a) & \boxed{k}\boxed{\overline{k+1}} & \xrightarrow{k} \boxed{k}\boxed{\bar{k}} & \xrightarrow{k} \boxed{k+1}\boxed{\bar{k}} & & \text{for } 1 \leq k < n \\
 & \boxed{\overline{k+1}}\boxed{\overline{k+1}} & \xrightarrow{k} \boxed{\overline{k+1}}\boxed{\bar{k}} & \xrightarrow{k} \boxed{\bar{k}}\boxed{\bar{k}} & & \text{for } 1 \leq k < n \\
(4.5b) & \boxed{n}\boxed{n} & \xrightarrow{n} \boxed{n}\boxed{\bar{n}} & \xrightarrow{n} \boxed{\bar{n}}\boxed{\bar{n}} & & \\
 & \boxed{\bar{1}}\boxed{\bar{1}} & \xrightarrow{0} \emptyset & \xrightarrow{0} \boxed{1}\boxed{1} & . &
\end{array}$$

Note that none of the elements satisfies (4.4).

For type $A_{2n-1}^{(2)}$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.6 and 6.7]. The elements are given by $\boxed{x}\boxed{y}$ with $x \leq y$ and $x, y \in \{1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\}$. The action of f_i for $i = 1, 2, \dots, n$ is the same as for the above $C_n^{(1)}$ crystal of level one, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\bar{1}$ (with appropriate reorderings).

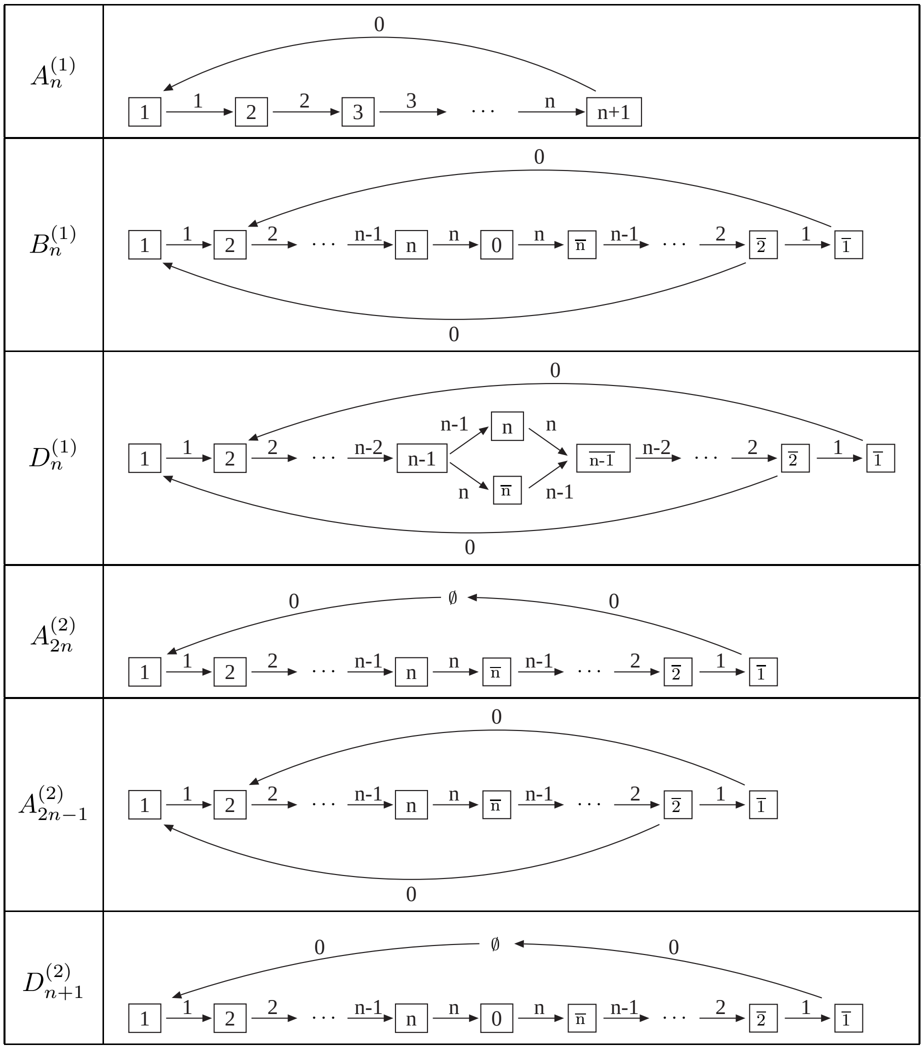

TABLE 1. Level one perfect crystals

The strings of length greater than one are the same as in (4.5a) and (4.5b). In addition there are the following 0-strings of length 2

$$\begin{array}{ccccc} \boxed{\overline{1}}\boxed{\overline{1}} & \xrightarrow{0} & \boxed{2}\boxed{\overline{1}} & \xrightarrow{0} & \boxed{2}\boxed{2} \\ \boxed{\overline{2}}\boxed{\overline{1}} & \xrightarrow{0} & \boxed{1}\boxed{\overline{1}} & \xrightarrow{0} & \boxed{1}\boxed{2} \\ \boxed{\overline{2}}\boxed{\overline{2}} & \xrightarrow{0} & \boxed{1}\boxed{\overline{2}} & \xrightarrow{0} & \boxed{1}\boxed{1} \end{array} . \tag{4.6}$$

The only elements fulfilling (4.4) are $\boxed{1}\boxed{\overline{1}}$, $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{\overline{1}}$, and $\boxed{2}\boxed{2}$ which belong to a 0-string and a 1-string of length two. It can be checked that setting $v(b) = 0$ for b one of these four elements guarantees the involutive condition of v.

For type $B_n^{(1)}$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.7 and 6.8]. It consists of the elements $\boxed{x\,|\,y}$ with $x \le y$ and $x, y \in \{1 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}\}$; $x = y = 0$ is excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $B_n^{(1)}$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\overline{1}$ (with appropriate reorderings).

The strings of length greater than one are those of equations (4.5a) and (4.6) and in addition the following n-string of length four

$$\boxed{n\,|\,n} \xrightarrow{n} \boxed{n\,|\,0} \xrightarrow{n} \boxed{n\,|\,\overline{n}} \xrightarrow{n} \boxed{0\,|\,\overline{n}} \xrightarrow{n} \boxed{\overline{n}\,|\,\overline{n}}\ . \tag{4.7}$$

The same four elements as for $A_{2n-1}^{(2)}$ satisfy (4.4) and again setting $v(b) = 0$ for these ensures the involutive property of v.

For type $D_n^{(1)}$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.8 and 6.9]. It consists of the elements $\boxed{x\,|\,y}$ with $x \le y$ and $x, y \in \{1 < 2 < \cdots < n, \overline{n} < \cdots < \overline{1}\}$, the cases $x = n, y = \overline{n}$ and $x = \overline{n}, y = n$ being excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D_n^{(1)}$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\overline{1}$ (with appropriate reorderings).

Again the strings of length greater than one are the same as in equations (4.5a) and (4.6) plus the following n-strings

$$\begin{array}{ccccc}
\boxed{n\,|\,n} & \xrightarrow{n} & \boxed{n\,|\,\overline{n-1}} & \xrightarrow{n} & \boxed{\overline{n-1}\,|\,\overline{n-1}} \\
\boxed{n-1\,|\,n-1} & \xrightarrow{n} & \boxed{n-1\,|\,\overline{n}} & \xrightarrow{n} & \boxed{\overline{n}\,|\,\overline{n}} \\
\boxed{n-1\,|\,n} & \xrightarrow{n} & \boxed{n-1\,|\,\overline{n-1}} & \xrightarrow{n} & \boxed{\overline{n}\,|\,\overline{n-1}}\ .
\end{array}$$

In addition to the four elements $\boxed{1\,|\,\overline{1}}$, $\boxed{1\,|\,2}$, $\boxed{2\,|\,\overline{1}}$, and $\boxed{2\,|\,2}$ also the elements $\boxed{\overline{n-1}\,|\,\overline{n-1}}$, $\boxed{n-1\,|\,\overline{n-1}}$, $\boxed{n\,|\,\overline{n-1}}$, and $\boxed{\overline{n}\,|\,\overline{n-1}}$ satisfy (4.4). The latter ones are contained in an $(n-1)$-string and an n-string. Setting $v(b) = 0$ for the first four elements and $v(b) = n$ for the last four elements ensures the involutive property of v.

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $D_{n+1}^{(2)}$ [10, Sections 1.9 and 6.10]. The elements of this crystal are $\emptyset$, $\boxed{x}$, and $\boxed{x\,|\,y}$ with $x, y \in \{1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}\}$ and $x \le y$; $x = y = 0$ is excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D_{n+1}^{(2)}$ as given in Table 1, and the action of f_0 is given by

$$\begin{array}{llll}
\emptyset & \xrightarrow{0} & \boxed{1} & \\
\boxed{x} & \xrightarrow{0} & \boxed{1\,|\,x} & \text{for } x \neq \overline{1} \\
\boxed{\overline{1}} & \xrightarrow{0} & \emptyset & \\
\boxed{x\,|\,\overline{1}} & \xrightarrow{0} & \boxed{x} & \text{for } x \neq 1
\end{array} \tag{4.8}$$

and undefined otherwise.

The strings of length greater than one are given by (4.5a), (4.7) and

$$\boxed{\overline{1}\,|\,\overline{1}} \xrightarrow{0} \boxed{\overline{1}} \xrightarrow{0} \emptyset \xrightarrow{0} \boxed{1} \xrightarrow{0} \boxed{1\,|\,1} \tag{4.9}$$

$$\boxed{\mathrm{n}} \xrightarrow{\mathrm{n}} \boxed{0} \xrightarrow{\mathrm{n}} \boxed{\overline{\mathrm{n}}} \tag{4.10}$$

There are no elements with property (4.4).

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $A_{2n}^{(2)}$ [10, Sec. 1.10 and 6.11]. The elements of this crystal are $\emptyset$, $\boxed{x}$, and $\boxed{x\,|\,y}$ with $x, y \in \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$ and $x \leq y$. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $A_{2n}^{(2)}$ as given in Table 1, and the action of f_0 is the same as in (4.8).

The strings of length greater than one are as in (4.5a) for $n \geq 2$, (4.5b) and (4.9). Again there are no elements with property (4.4). $\square$

Remark 5. Suppose $\mathfrak{g}$ is of type $A_{n-1}^{(1)}$ in Lemma 4. The function v amounts to a canonical choice of a simple root i among those such that the given element admits e_i. Consider $b \in \mathbb{B}(\Lambda_r)$ such that $b \neq u_{\Lambda_r}$. In addition to the realization of the crystal $\mathbb{B}(\Lambda_r)$ by the space of homogeneous paths using the crystal given in the proof of Lemma 4, one may also consider the realization in [2] by n-regular partitions. Suppose λ is the partition corresponding to b. Then up to the Dynkin diagram automorphism that sends $r+i$ to $r-i$ modulo n, the choice of violation v corresponds to the corner cell of λ that is in the rightmost column of λ. This choice of corner cell is used in [15] to define the smallest Demazure crystal of $\mathbb{B}(\Lambda_r)$ containing b.

5. Inhomogeneous paths

Theorem 6. *Let $\mathfrak{g}$ be as in Theorem 3, and B, Λ, and B_0 be as in* (3.5)*. Suppose in addition that for all $1 \leq j \leq L$ and $b \in B_j$, if $b \otimes b_0 \mapsto b_0' \otimes b'$ under the local isomorphism $B_j \otimes B_0 \to B_0 \otimes B_j$ and $e_0(b \otimes b_0) = e_0(b) \otimes b_0$ then $e_0(b_0' \otimes b') = e_0(b_0') \otimes b'$. Then*

$$\Pi\left(\mathrm{ch}(B \otimes u_\Lambda)\right) = \mathrm{ch}(B \otimes \mathbb{B}(\Lambda)). \tag{5.1}$$

Proof. It is enough to verify the hypotheses of Theorem 3, applied to $B \otimes u_\Lambda \subset B \otimes \mathbb{B}(\Lambda)$. $B \otimes \mathbb{B}(\Lambda)$ is isomorphic to a direct sum of irreducible integrable highest weight modules by Theorem 2. $B \otimes u_\Lambda$ is obviously closed under the e_i. It follows from [11, Lemma 1] that $B \otimes u_\Lambda$ generates $B \otimes \mathbb{B}(\Lambda)$. To check the third condition of Theorem 3, let $b \in B$ and $i \in I$ be such that $\epsilon_i(b \otimes u_\Lambda) > 0$. Then $\epsilon_i(b) > \phi_i(u_\Lambda) = \langle \alpha_i^\vee, \Lambda \rangle$. This implies that the i-string of $b \otimes u_\Lambda$ inside $B \otimes \mathbb{B}(\Lambda)$, consists of vectors of the form $b' \otimes u_\Lambda$ where $b' \in B$.

Finally, Lemma 1 with B replaced by $B \otimes B_0$ guarantees that the affine weight function on $B \otimes \mathbb{B}(\Lambda)$ determined by its value on highest weight vectors, agrees on the subset $B \otimes u_\Lambda$ with the function $\mathrm{wt}(b) = \mathrm{af}(\mathrm{wt}'(b)) - E_{B,B_0}(b \otimes b_0) a_0^{-1} \delta$ where $\mathrm{wt}' : B \to P_{cl}$ is the original weight function. $\square$

Remark 7. Observe that even without the extra hypothesis on the action of e_0 in Theorem 6, one obtains a bosonic formula. The extra condition is only needed to show that the energy function $b \mapsto E_{B,B_0}(b \otimes b_0)$ gives rise to the correct affine

weight for all elements of the form $b \otimes u_\Lambda$ and not just on the highest weight vectors. Perhaps this extra condition is always a consequence of the other hypotheses.

Now the formula (5.1) is written more explicitly. Let $m \in \mathbb{Z}$ and $\Lambda, \Lambda' \in \mathrm{af}(P^+_{cl})$ be of level ℓ. A formula equivalent to (5.1) is obtained by taking the coefficient of $\mathrm{ch}\mathbb{V}(\Lambda' - m a_0^{-1}\delta)$ on both sides:

$$[q^m]K(B, \Lambda, \Lambda', B_0)(q) = \sum_{(w,b)\in\mathcal{S}} (-1)^w$$

where $\mathcal{S}$ is the set of pairs $(w, b) \in W \times B$ such that

$$w^{-1}(\Lambda' + \rho) - m a_0^{-1}\delta - \rho = \mathrm{wt}(b \otimes u_\Lambda). \tag{5.2}$$

Let M be the sublattice of $\overline{P}$ given by the image under ν of the $\mathbb{Z}$-span of the orbit $\overline{W}\theta^\vee$. Let $T \subset GL(\mathfrak{h}^*)$ be the group of translations by the elements of M, where $t_\alpha \in T$ is translation by $\alpha \in M$. Then $W \cong T \rtimes \overline{W}$ and $r_0 = t_{\nu(\theta^\vee)} r_\theta$. For $\alpha \in M$ and $\Lambda \in P$ of level ℓ, one has [7, (6.5.2)]

$$t_\alpha(\Lambda) = \Lambda + \ell\alpha - ((\Lambda|\alpha) + \tfrac{1}{2}|\alpha|^2\ell)\delta. \tag{5.3}$$

The action of $\tau \in \overline{W}$ on the level ℓ weight Λ is given by

$$\tau(\Lambda) = \tau(\overline{\Lambda} + \ell\Lambda_0) = \tau(\overline{\Lambda}) + \ell\Lambda_0.$$

Now $\rho = h^\vee\Lambda_0 + \overline{\rho}$ where $h^\vee$ is the dual Coxeter number and $\overline{\rho}$ is the half-sum of the positive roots in $\overline{\mathfrak{g}}$.

Recall that $\overline{W}$ leaves δ invariant. In (5.2) write $w = t_\alpha\tau$ where $\tau \in \overline{W}$ and $\alpha \in M$, obtaining

$$\begin{aligned}
\mathrm{wt}(b \otimes u_\Lambda) =& \tau^{-1}t_{-\alpha}(\Lambda' + \rho) - m a_0^{-1}\delta - \rho \\
=& - m a_0^{-1}\delta - \rho + \tau^{-1}\big\{\Lambda' + \rho - (\ell + h^\vee)\alpha \\
& - \{(\Lambda' + \rho| - \alpha) + \tfrac{1}{2}|\alpha|^2(\ell + h^\vee)\}\delta\big\} \\
=& \ell\Lambda_0 - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} + \overline{\rho} - (\ell + h^\vee)\alpha) \\
& + \big\{-m a_0^{-1} + (\overline{\Lambda'} + \overline{\rho}|\alpha) - \tfrac{1}{2}|\alpha|^2(\ell + h^\vee)\big\}\delta
\end{aligned}$$

Since both sides are weights of level ℓ, by equating coefficients of δ and projections into $\overline{P}$, one obtains the equivalent conditions

$$\overline{\mathrm{wt}(b)} = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} - (\ell + h^\vee)\alpha + \overline{\rho}) \tag{5.4}$$

and

$$a_0^{-1}E(b) = a_0^{-1}m - (\overline{\Lambda'} + \overline{\rho}|\alpha) + \tfrac{1}{2}|\alpha|^2(\ell + h^\vee). \tag{5.5}$$

Therefore one has the equality

$$K(B, \Lambda, \Lambda', B_0)(q) = \sum_{\tau\in\overline{W}}\sum_{\alpha\in M}\sum_{b\in B}(-1)^\tau q^{E(b) + a_0(\overline{\Lambda'} + \overline{\rho}|\alpha) - \frac{1}{2}a_0|\alpha|^2(\ell + h^\vee)} \tag{5.6}$$

where $b \in B$ satisfies

$$\mathrm{wt}(b) = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} - (\ell + h^\vee)\alpha + \overline{\rho}).$$

6. Type A

6.1. **Conjecture of** [3]**.** For simplicity let us assume that $\mathfrak{g}$ is of untwisted affine type, where $a_0 = 1$ and $(\overline{\rho}|\theta) = h^\vee - 1$ [7, Ex. 6.2].

Let $\Lambda \in P$ be a weight of level ℓ but not necessarily dominant. Consider the weight $\Lambda + \rho$. If it is regular (not fixed by any $w \in W$) then there is a unique $w \in W$ such that $w(\Lambda + \rho) \in P^+$. It follows from the definition of Π that

$$\Pi\, e^\Lambda = \begin{cases} (-1)^w \mathrm{ch}\mathbb{V}(w(\Lambda+\rho)-\rho) & \text{if } \Lambda+\rho \text{ is } W\text{-regular and} \\ & w(\Lambda+\rho) \in P^+ \\ 0 & \text{if } \Lambda+\rho \text{ is not } W\text{-regular.} \end{cases} \tag{6.1}$$

Then for all $i \in I$,

$$-\Pi\, e^\Lambda = \Pi\, e^{r_i(\Lambda+\rho)-\rho}. \tag{6.2}$$

Suppose $i \neq 0$. Then

$$\begin{aligned} r_i(\Lambda+\rho) - \rho &= (\ell + h^\vee)\Lambda_0 + r_i(\overline{\Lambda} + \overline{\rho}) - (h^\vee \Lambda_0 + \overline{\rho}) \\ &= \ell\Lambda_0 - \alpha_i + r_i(\overline{\Lambda}). \end{aligned}$$

For $i = 0$, recall that

$$r_0 = t_{\nu(\theta^\vee)} r_\theta = t_\theta r_\theta = r_\theta t_{-\theta}.$$

Then

$$\begin{aligned} t_{-\theta}(\Lambda+\rho) &= \Lambda + \rho - (\ell+h^\vee)\theta + \{(\Lambda+\rho|\theta) - \tfrac{1}{2}|\theta|^2(\ell+h^\vee)\}\delta \\ &= (\ell+h^\vee)\Lambda_0 + \overline{\rho} + \overline{\Lambda} - (\ell+h^\vee)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta \end{aligned}$$

and

$$\begin{aligned} r_0(\Lambda+\rho) - \rho &= r_\theta\big\{(\ell+h^\vee)\Lambda_0 + \overline{\rho} + \overline{\Lambda} - (\ell+h^\vee)\theta \\ &\quad + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta\big\} - \rho \\ &= (\ell+h^\vee)\Lambda_0 + \overline{\rho} - \langle\theta^\vee, \overline{\rho}\rangle\theta + r_\theta(\overline{\Lambda}) \\ &\quad + (\ell+h^\vee)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta - (h^\vee\Lambda_0 + \overline{\rho}) \\ &= \ell\Lambda_0 + r_\theta(\overline{\Lambda}) + (\ell+1)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta. \end{aligned}$$

Now let $\mathfrak{g}$ be of type $A^{(1)}_{n-1}$. Let $\overline{P}$ be identified with the subspace of $\mathbb{Z}^n$ given by vectors with sum zero.

For $\alpha \in \overline{P}$ define the Demazure operator $\overline{\Pi}$ to be the linear operator on $\mathbb{Z}[\overline{P}]$ such that

$$s_\alpha := \overline{\Pi}(e^\alpha) = \overline{J}^{-1}(e^{\overline{\rho}})\overline{J}(e^{\overline{\rho}+\alpha})$$

where $\overline{J} = \sum_{\tau \in \overline{W}} (-1)^\tau \tau$. Let $q = e^{-\delta}$. Then for $\alpha \in \overline{P}$,

$$-\Pi\, e^{\ell\Lambda_0} e^\alpha = \begin{cases} \Pi\, e^{\ell\Lambda_0} e^{r_i(\alpha)-\alpha_i} & \text{for } i \neq 0 \\ \Pi\, e^{\ell\Lambda_0} e^{r_\theta(\alpha)+(\ell+1)\theta} q^{\ell+1-(\alpha|\theta)} & \text{for } i = 0. \end{cases} \tag{6.3}$$

These equations express the q-equivalence in [3]. Let $\mathbb{Z}^n$ have standard basis $\{\epsilon_i \mid 1 \leq i \leq n\}$ and $\overline{P}$ be the subspace of $\mathbb{Z}^n$ orthogonal to the vector $\sum_{i=1}^n \epsilon_i$. Then $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, $\theta = \epsilon_1 - \epsilon_n$, $(\cdot|\cdot)$ is the ordinary dot product in $\mathbb{Z}^n$, and $\overline{W}$ is the symmetric group on n letters acting on the coordinates of $\mathbb{Z}^n$. Since $\Pi \circ \overline{\Pi} = \Pi$ and $\overline{\Pi}$ is $\mathbb{Z}\Lambda_0$-linear, one may replace every term e^α by $s_\alpha := \overline{\Pi} e^\alpha$ in (6.3). Define the map $\mathbb{Z}[\overline{P}]^{\overline{W}}[q] \to \mathbb{Z}[\overline{P}]^{\overline{W}}[q]$ given by $s_\alpha \mapsto \Pi(e^{\ell\Lambda_0+\alpha})e^{-\ell\Lambda_0}$.

Define $f \equiv g$ in $\mathbb{Z}[\overline{P}]^{\overline{W}}[q]$ by the condition that the above linear map sends f and g to the same element. With this definition, we have

$$-s_\alpha \equiv \begin{cases} s_{(\alpha_1,\dots,\alpha_{i+1}-1,\alpha_i+1,\dots,\alpha_n)} & \text{for } i \neq 0 \\ s_{(\ell+1+\alpha_n,\alpha_2,\dots,\alpha_{n-1},-1-\ell+\alpha_1)} q^{\ell+1-\alpha_1+\alpha_n} & \text{for } i = 0. \end{cases} \tag{6.4}$$

It is not hard to see that this recovers the q-equivalence of Schur functions given in [3].

6.2. **Bosonic conjecture of** [20, (9.2)]**.** In this section it is assumed that $\mathfrak{g}$ is of type $A^{(1)}_{n-1}$, $\Lambda = \ell\Lambda_0$, and the tensor factors B_j are perfect crystals of the form B^{k_j,ℓ_j} in the notation of [10] with $\ell_j \leq \ell$ for all j. By restriction to $U_q(\overline{\mathfrak{g}})$, B_j is the crystal of the irreducible integrable $U_q(\overline{\mathfrak{g}})$-module of highest weight $\ell_j\overline{\Lambda}_{k_j}$. In this case B_0 is not needed. To see this, recall that B_j can be realized as the set of column-strict Young tableaux of the rectangular shape having k_j rows and ℓ_j columns with entries in the set $\{1, 2, \dots, n\}$. In [19] the P_{cl}-weighted I-crystal structure on the perfect crystals $B^{k,\ell}$ is computed explicitly. In particular, if $b \in B_j$ is a tableau then $\epsilon_0(b)$ is at most the number of ones in the tableau b, which is at most ℓ_j by column-strictness. Therefore $b \otimes u_{\ell\Lambda_0}$ never admits e_0. Thus the energy function E_B of (3.2) has the property that for any $b \in B = B_L \otimes \cdots \otimes B_1$ such that $e_0(b \otimes u_{\ell\Lambda_0}) = e_0(b) \otimes u_{\ell\Lambda_0}$, one has $E_B(e_0(b)) = E_B(b) - 1$. Thus one obtains the bosonic formula in this case.

Since $\mathfrak{g}$ is of type $A^{(1)}_{n-1}$, $a_0 = 1$ and $h^\vee = n$. Take $\Lambda = \Lambda' = \ell\Lambda_0$ in (5.6). The lattice M is given by the root lattice $\overline{Q}$ of $\overline{\mathfrak{g}}$, which may be realized by $\{\beta \in \mathbb{Z}^n \mid \sum_{i=1}^n \beta_i = 0\}$. Let $B_{\tau,\beta}$ be the set of paths $b \in B$ of weight $-\overline{\rho} + \tau^{-1}(-(\ell+n)\beta + \overline{\rho})$. Then

$$\begin{aligned} K(B, \ell\Lambda_0, \ell\Lambda_0)(q) &= \sum_{\tau \in \overline{W}} \sum_{\beta \in M} \sum_{b \in B_{\tau,\beta}} (-1)^\tau q^{E_B(b) + (\overline{\rho}|\beta) - \frac{1}{2}|\beta|^2(\ell+n)} \\ &= \sum_{\tau \in \overline{W}} \sum_{\beta \in M} \sum_{b \in B_{\tau,\beta}} (-1)^\tau q^{E_B(b) - \sum_{i=1}^n \{\frac{1}{2}(\ell+n)\beta_i^2 + i\beta_i\}}. \end{aligned}$$

Notice that $\sum_{b \in B_{\tau,\beta}} q^{E_B(b)}$ is (up to an overall factor) the $q \to 1/q$ form of the supernomial S of ref. [20] so that $K(B, \ell\Lambda_0, \ell\Lambda_0)(q)$ equals the left-hand side of [20, (9.2)] up to an overall power of q. This shows that the left-hand side of [20, (9.2)] is indeed the generating function of level-ℓ restricted paths. To establish the equality [20, (9.2)] it remains to prove that also the right-hand side equals the generating function of level-restricted paths.

6.3. **Identities for level one and level zero.** As in the previous section let $\mathfrak{g}$ be of type $A^{(1)}_{n-1}$ and assume that $B = B^{k_L,1} \otimes \cdots \otimes B^{k_1,1}$. Fix $\ell = 1$ and $\Lambda, \Lambda' \in P^+_{cl}$ weights of level 1. It is easy to verify that $\mathcal{P}(B, \Lambda, \Lambda')$ consists of at most one element p. Choose B, Λ, Λ' such that $p \in \mathcal{P}(B, \Lambda, \Lambda')$ exists. Then by (3.6) and (5.6) we find that

$$\sum_{\tau \in \overline{W}} \sum_{\beta \in M} \sum_{b \in B_{\tau,\beta,\Lambda,\Lambda'}} (-1)^\tau q^{E(b) - \sum_{i=1}^n \{\frac{n+1}{2}\beta_i^2 + i\beta_i\}} = q^{E(p)} \tag{6.5}$$

where $B_{\tau,\beta,\Lambda,\Lambda'}$ is the set of paths $b \in B$ of weight $-\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda}' - (n+1)\beta + \overline{\rho})$.

A similar formula exists for $\ell = 0$:

$$\sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{b\in B_{\tau,\beta}}(-1)^{\tau}q^{E(b)-\sum_{i=1}^{n}\{\frac{n}{2}\beta_i^2+i\beta_i\}}=\delta_{B,\emptyset} \tag{6.6}$$

where $B_{\tau,\beta}$ is the set of paths $b \in B$ of weight $-\overline{\rho}+\tau^{-1}(-n\beta+\overline{\rho})$. The right-hand side is the generating function of paths in B of level zero since there are no level zero restricted paths unless B is empty. However, the arguments of Sections 4 and 5 do not imply that also the left-hand side is the generating function of level zero paths since it was assumed in the proof of Theorem 3 that the level of the crystals B_j does not exceed ℓ. We have assumed that $B_j = B^{k_j,1}$ which are crystals of level one. However, it is possible to define a sign-reversing involution directly on $B = B^{k_L,1} \otimes \cdots \otimes B^{k_1,1}$ without using the crystal isomorphisms that are used in the proof of Theorem 3. Let $b \in B$. There exists at least one $0 \le i \le n$ such that $e_i(b_1)$ is defined. Define $v(b) = \min\{i | e_i(b_1) \text{ is defined}\}$ which has the property that $v(b) = v(\Phi_i(b))$ where as before $\Phi_i = s_i e_i$. Hence define the involution $\Phi(b) = \Phi_{v(b)}(b)$. It is again sign-reversing and has no fixed points when $B \neq \emptyset$. This proves that the left-hand side of (6.6) is the generating function of level 0 restricted paths.

Equation (6.5) was conjectured in [20, 21]. For $n = 2$ identity (6.6) follows from the q-binomial theorem, for $n = 3$ it was proven in [1, Proposition 5.1] and for general n it was conjectured in [21].

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